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Nonorthodox Guidance Law Development Approach for Intercepting Maneuvering Targets

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Two aspects of a new mindset are applied to interceptor guidance: 1) the mathematical formulation of an interception scenario against maneuverable targets and 2) the relationship between the estimation process and optimal guidance law design. The interception of a maneuverable antisurface missile is formulated as a zero-sum pursuit evasion game. The perfect information game solution, which guarantees a robust hit-to-kill homing accuracy, requires, among other things, the knowledge of the target maneuver. This variable cannot be directly measured and has to be estimated based on noisy measurements. The greatest error source in the estimation of time-varying target maneuvers is the inherent delay due to the convergence time of the process. The estimation process is modeled as a pure information delay. Such a delay is partially compensated by a new guidance law based on the solution of a delayed information pursuit–evasion game. The new approach represents a potential breakthrough in guidance law design predicting reduced miss distances and robustness even in stressing interception environments. The improved accuracy is confirmed by a set of Monte Carlo simulations in a ballistic missile interception scenario with noisy measurements.

I. Introduction

IN contrast to the impressive technological progress achieved by the guided missile community, the concepts of interceptor guidance law development have remained (unfortunately) conservative. This conservative mindset can be attributed to that until recently the target of an interceptor missile was an inhabited aircraft, against which the missile had substantial advantage in speed, maneuverability, and agility. Moreover, miss distances of the order of a few meters (compatible with the lethal radius of the missile warhead) were considered admissible. Thus, the effort to develop improved guidance laws may have seemed to be unnecessary.

The 1991 Gulf War introduced a new type of target, namely, the tactical ballistic missile (TBM), able to carry nonconventional warheads. Successful interception of a TBM, much less vulnerable than an aircraft, requires a very small miss distance or even a direct hit, creating a new challenge for the guided missile community. Several defense systems against ballistic missiles are currently in development to meet the threat. All of them have been designed against ballistic targets, flying predictable trajectories. The available advanced technology allowed these defense systems (ARROW, PAC-3, THAAD) to demonstrate, in spite of using conventional guidance and estimation concepts, an excellent homing accuracy (sometimes even a kinetic hit to kill) against such nonmaneuvering targets.[1–3]

Although known TBMs were not designed to maneuver, due to their high reentry velocity they have a substantial maneuverability potential. Moreover, this great maneuverability potential can be made applicable by a modest technical effort. The same is true for future high-speed antiship or cruise missiles. Paradoxically, the successful current development of antiballistic missile defense systems can motivate the development of a new generation of highly maneuverable antisurface missiles. Against such threats, interceptor missiles will have only a marginal maneuverability advantage. Recent simulation studies of anticipated antimissile defense scenarios clearly indicate that currently used guidance laws and estimation techniques are unable to guarantee an adequate homing accuracy for a hit to kill in the interception of the expected highly maneuvering targets.[4–6]

The great majority of advanced missile guidance laws in current use were developed by using a linearized kinematical model and solving a linear quadratic optimal control problem, where the limited maneuver potential of the interceptor was not explicitly taken into account.[7] As is well known, the optimal control concept requires information on the current and future target maneuvers. In most cases, for the sake of simplicity, constant target maneuvers were assumed.

There is a basic deficiency in formulating the interception of a maneuverable target as an optimal control problem. Target maneuvers are independently controlled; thus, future target maneuvers cannot be predicted. As a consequence, the optimal control formulation of such problems is not appropriate. The mathematical framework for analyzing conflicts controlled by two independent agents is in the realm of dynamic games. Thus, the scenario of intercepting a maneuverable target has to be formulated as a zero-sum pursuit–evasion game. The game solution provides simultaneously the optimal pursuer strategy (the missile's guidance law), the optimal evader strategy (the worst target maneuver), and the value of the game (the miss distance guaranteed to the interceptor missile as well as to the maneuvering target, by using the respective optimal strategies).

Although the concept of such a formulation dates back to the 1950s and was published in the seminal book by Isaacs in 1965,[8] the guided missile industry has not fully recognized the potential involved in it. Nevertheless, the idea has raised some academic interest, as is evident in the open literature. Whereas a linear quadratic game formulation with an ideal dynamic model lead to proportional navigation[9] as an optimal guidance law, a more realistic analysis recognized that the controls are bounded and interceptor missile dynamics should be represented at least by a first-order transfer function. Such an analysis,[10] limited to a planar scenario, was published in 1979 and was extended later to three dimensions.[11] It yielded a game optimal guidance law that explicitly accounts for the limited interceptor maneuverability, and the assumption of ideal target maneuver dynamics (the worst case for the interceptor) eliminated the need of knowing the actual target maneuver. This differential game guidance law (DGL/0) provides robustness with respect to the type of target maneuver but cannot guarantee a zero miss distance. If, in

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this perfect information zero-sum pursuit–evasion game formulation, first-order (nonideal) target maneuver dynamics are assumed and the actual target maneuver is available, an improved guidance law can be used.[12] This guidance law, DGL/1, guarantees from the majority of initial conditions of practical interest a zero-miss distance, if the agility (defined as the maximum acceleration divided by the first-order time constant) of the interceptor is superior to that of the target.

A detailed comparison study,[13] based on extensive simulations, clearly shows the superiority of interceptor guidance laws derived from a differential game formulation over those obtained using optimal control theory. In spite of the results of this comparison, and a subsequent paper advocating this approach for guidance law synthesis of future interceptor missiles,[14] differential game guidance laws have not been adopted by the missile industry. This conservative attitude can be explained by that the already existing interceptor missiles had a sufficient maneuverability advantage over the presumed inhabited aircraft targets. Moreover, applying the optimal pursuer strategy of the perfect information game as the interceptor's guidance law using a typical estimator yields very disappointing results. The miss distance is never zero, and there is a high sensitivity to the structure of the (unknown) target maneuver.[15]

The poor performance of perfect information guidance laws in a noisy environment can be attributed to the unjustified reliance on the certainty equivalence property[16] that states that the optimal control law for a stochastic control problem is the same as the optimal control law for the associated deterministic (certainty equivalent) problem. The validity of the certainty equivalence property was proved for linear optimal control problems with unbounded control, quadratic cost function and Gaussian noise with a strictly classical information pattern, where the controller continously retains knowledge of all past outputs and controls. Note that the certainty equivalence property has never been proved for realistic missile guidance problems, characterized by bounded control, non-Gaussian random target maneuvers, and saturated state variables.

For problems with a strictly classical information pattern, the state estimator can be designed independently of the control law even if the certainty equivalence property does not hold. However, the stochastic optimal control depends on the conditional probability density function of the estimated variables resulting from the solution of the filtering problem.[17] Unfortunately, this very important observation has not yet been applied in any guided missile design. Following this idea, an intuitive attempt was made[18] to derive a new guidance law that takes into account a simplified model of the estimation process. Because simulation results indicated that the largest error source is the inherent delay in estimating the target acceleration, the entire estimation process was roughly approximated by a pure time delay in the (otherwise ideal) measurement of the target accelerationl.[15,18] In a recent paper,[19] a rigorous analytical solution of a delayed information pursuit–evasion game was presented, confirming the validity of the earlier intuitive approach.

The objective of this paper is to apply the delay-compensating guidance law against a highly maneuvering antisurface missile in an interception scenario with noise-corrupted measurements. Guidance law derivation based on explicitly considering the imperfection of the estimation process is, in fact, a pioneering endeavor, representing a potential breakthrough in interceptor guidance law design.

After the deterministic formulation of the problem in Sec. II, the perfect information game solution is outlined. It is followed by a discussion on the implementation of the resulting guidance law with noisy measurements and the modeling of the estimation process. In Sec. IV, the delayed information pursuit–evasion game is formulated, and its rigorous analytical solution, detailed in Ref. 19, is described. The actual homing performance of the new guidance law, tested by Monte Carlo simulations in a realistic environment of noise-corrupted measurements using two different types of estimators, is presented in Sec. V.

II. Problem Statement

The interception scenario of a maneuvering antisurface missile is formulated as a linear zero-sum pursuit–evasion game with bounded controls. The deterministic game analysis is based on the following set of simplifying assumptions:

1) The engagement between the interceptor (pursuer) and the antisurface missile (evader) takes place in a plane.

2) Both missiles have constant speeds V_j and bounded lateral accelerations, $|a_j| < (a_j)^{\max}$, $j = E, P$.

3) The maneuvering dynamics of both missiles can be approximated by first-order transfer functions with time constants τ_P and τ_E.

4) The trajectories of both missiles can be linearized along the initial line of sight.

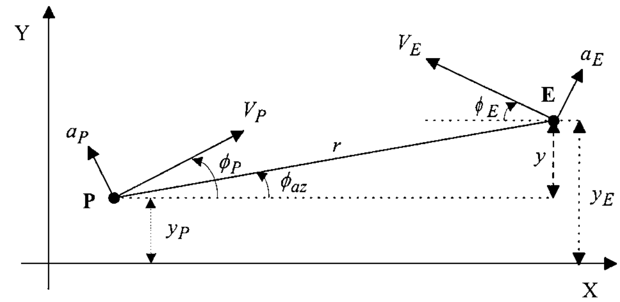

Fig. 1 End-game geometry.

In Fig. 1, a schematic view of the end-game geometry is shown. Note that the respective velocity vectors of the missiles are generally not aligned with the reference line of sight. The aspect angles ϕ_P and ϕ_E are, however, small. Thus, the approximations $\cos(\phi_i) \approx 1$ and $\sin(\phi_i) \approx (\phi_i)$, $i = P, E$, are uniformly valid and coherent with assumption 4. Moreover, based on assumptions 2 and 4, the final time of the interception can be computed for any given initial range X_0 of the end game:

$$t_f = X_0/(V_P + V_E) \tag{1}$$

allowing to define the time to go by

$$t_{go} = t_f - t \tag{2}$$

The state vector in the equations of relative motion normal to the reference line is

$$X^T = (x_1, x_2, x_3, x_4) = (y, \dot{y}, a_E, a_P) \tag{3}$$

where

$$y(t) \triangleq y_E(t) - y_P(t) \tag{4}$$

The corresponding equations of motion and the respective initial conditions are

$$\dot{x}_1 = x_2, \qquad x_1(0) = 0 \tag{5}$$

$$\dot{x}_2 = x_3 - x_4, \qquad x_2(0) = V_E\phi_{E_0} - V_P\phi_{P_0} \tag{6}$$

$$\dot{x}_3 = \left(a_E^c - x_3\right)/\tau_E, \qquad x_3(0) = 0 \tag{7}$$

$$\dot{x}_4 = \left(a_P^c - x_4\right)/\tau_P, \qquad x_4(0) = 0 \tag{8}$$

where a_E^c and a_P^c are the commanded lateral accelerations of E and P, respectively:

$$a_E^c = (a_E)^{\max} \boldsymbol{v}, \qquad |\boldsymbol{v}| \le 1 \tag{9}$$

$$a_P^c = (a_P)^{\max} \boldsymbol{u}, \qquad |\boldsymbol{u}| \le 1 \tag{10}$$

The nonzero initial conditions $V_E\phi_{E_0}$ and $V_P\phi_{P0}$ represent the respective initial velocity component not aligned with the initial (reference) line of sight. By assumption 4, these components are small compared to the components along the line of sight.

The set of Eqs. (5–8) can be written in a compact form as a linear, time-invariant, vector differential equation:

$$\dot{X} = AX + B\boldsymbol{u} + C\boldsymbol{v} \tag{11}$$

The natural cost function of the game is the miss distance

$$J = |DX(t_f)| = |x_1(t_f)| \tag{12}$$

where

$$D = (1, 0, 0, 0) \tag{13}$$

The problem involves two nondimensional parameters of physical significance: One is the pursuer–evader maximum maneuverability ratio

$$\mu \triangleq (a_P)^{\max} \big/ (a_E)^{\max} \tag{14}$$

and the other is the ratio of the evader–pursuer time constants

$$\varepsilon \triangleq \tau_E/\tau_P \tag{15}$$

The vector differential equation (11) can be reduced to a scalar one by using the transformation

$$z(t) = D\Phi(t_f, t)X(t) \tag{16}$$

where $\Phi(t_f, t)$ is the transition matrix of the original homogeneous system of Eq. (11). The new state variable $Z(t)$ is the zero-effort miss distance, a well-known term in guidance analysis.[7] It is the miss distance that results if both players do not apply any further acceleration commands. The calculation of $Z(t)$ is based on the homogenous solution of Eq. (11) and, as such, is a function of the current state $X(t)$.

For the sake of generality, nondimensional variables are defined. The independent variable is the normalized time to go

$$\theta \triangleq (t_f - t)/\tau_P, \qquad \theta(0) = t_f/\tau_P = \theta_0 \tag{17}$$

The nondimensional state variable is the normalized zero-effort miss distance

$$Z(\theta) \triangleq \frac{z(t)}{\tau_P^2 (a_E)^{\max}} = \frac{x_1 + x_2(\tau_P\theta) + x_3\tau_E^2\psi(\theta/\varepsilon) - x_4\tau_P^2\psi(\theta)}{\tau_P^2 (a_E)^{\max}} \tag{18}$$

where

$$\psi(\alpha) \triangleq e^{-\alpha} + \alpha - 1 \tag{19}$$

The normalized initial condition is

$$Z(\theta_0) \triangleq Z_0 = \frac{(V_E\phi_{E0} - V_P\phi_{P0})\theta_0}{\tau_P (a_E)^{\max}} \tag{20}$$

The explicit expression (18) imbeds the assumption of perfect information, that is, that all of the original state variables (x_1 and x_2, as well as the lateral accelerations x_3 and x_4) are known to both players. When the nondimensional variables are used, the normalized game dynamics becomes

$$\frac{dZ}{d\theta} = \mu\psi(\theta)\boldsymbol{u} - \varepsilon\psi\left(\frac{\theta}{\varepsilon}\right)\boldsymbol{v}, \qquad Z(\theta_0) = Z_0 \tag{21}$$

Note that the game dynamics are independent of the state variables of the game because $Z(t)$ itself is based on the homogeneous solution of Eq. (11).

The nondimensional payoff function is the normalized miss distance,

$$J = |Z_f| = |Z(\theta = 0)| \tag{22}$$

to be minimized by the pursuer and maximized by the evader.

III. Guidance Law Implementation

A. Perfect Information Game Solution

The solution of the perfect information version of this game[12] is described briefly. The necessary conditions of optimality provide the optimal control strategies of the players as

$$\boldsymbol{u}^*(Z, \theta) = \boldsymbol{v}^*(Z, \theta) = \mathrm{sign}\{Z(\theta)\}, \qquad Z(\theta) \neq 0 \tag{23}$$

Substituting Eq. (23) into Eq. (21) yields the dynamics along candidate optimal trajectories,

$$\frac{dZ^*}{d\theta} = \Gamma(\theta)\,\mathrm{sign}\{Z(\theta)\} \tag{24}$$

with

$$\Gamma(\theta) = \mu\psi(\theta) - \varepsilon\psi(\theta/\varepsilon) \tag{25}$$

When Eq. (24) is intergrated, a family of regular optimal trajectories is generated. If this family does not fill the entire nondimensional game space (Z, θ), the game solution yields a decomposition into two regions (Fig. 2). In the regular region, denoted as D_1, the optimal strategies are of bang–bang type given by Eq. (23), and the value of the game is a unique function of the initial conditions. The boundaries of this region are the pair of optimal trajectories Z_+^* and Z_-^*, which reach the θ axis, $Z = 0$, tangentially at the normalized time to go θ_s, as shown in Fig. 2, where $\theta_s = \theta_s(\mu, \varepsilon)$ is the nonvanishing solution of the equation $\Gamma(\theta) = 0$. The other (singular) region, enclosed by the boundaries Z_+^* and Z_-^* and denoted as D_0, has several particular features. All of the trajectories starting in D_0 must go through the point $(Z = 0, \theta = \theta_s)$ called the throat, where D_0 terminates. Therefore, every trajectory in this region can be considered as optimal, and the optimal strategies cannot be uniquely determined. They can be selected arbitrarily, such as a linear strategy suggested by Ref. 10 or even a bang–bang-type strategy such as Eq. (23). The value of the game for the entire region, obtained by integrating $\Gamma(\theta)$ between $\theta = 0$ and θ_s and using the very definition of θ_s, is constant:

$$J_0^* = (1 - \varepsilon)\psi(\theta_s) - (\mu - 1)\theta_s^2 \big/ 2 \triangleq M_s(\mu, \varepsilon) \tag{26}$$

When a trajectory that started in D_0 reaches the throat, the evader must select the direction of its maximal maneuver (either to the right or to the left), and the pursuer has to follow it. Therefore, the optimal evasive maneuver that guarantees the value of $M_s(\mu, \varepsilon)$ is a maximal maneuver in a fixed direction for the duration of at least $\theta_s(\mu, \varepsilon)$. The values of $\theta_s(\mu, \varepsilon)$ and $M_s(\mu, \varepsilon)$ are shown in Figs. 3 and 4, clearly showing the dependence on these physical parameters. These results are of great importance, because for the majority of important practical cases, the initial conditions of an interception are in D_0.

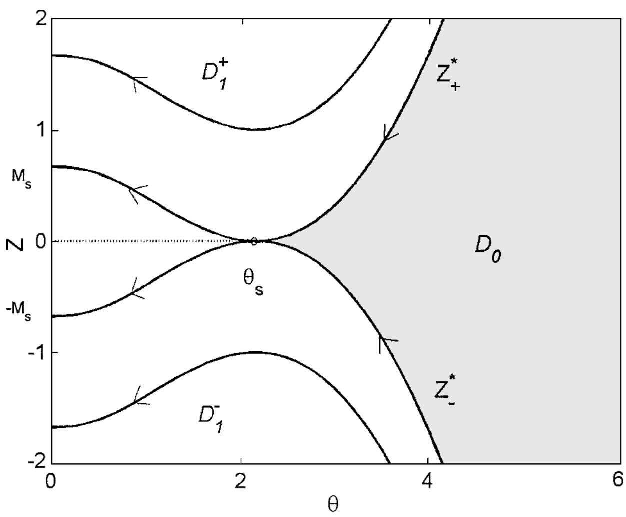

Fig. 2 Decomposition of the game space.

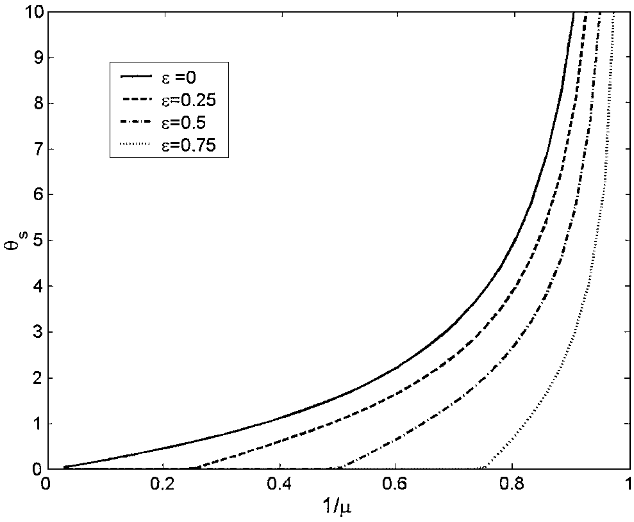

Fig. 3 Critical normalized time to go $\theta_s(\mu, \varepsilon)$.

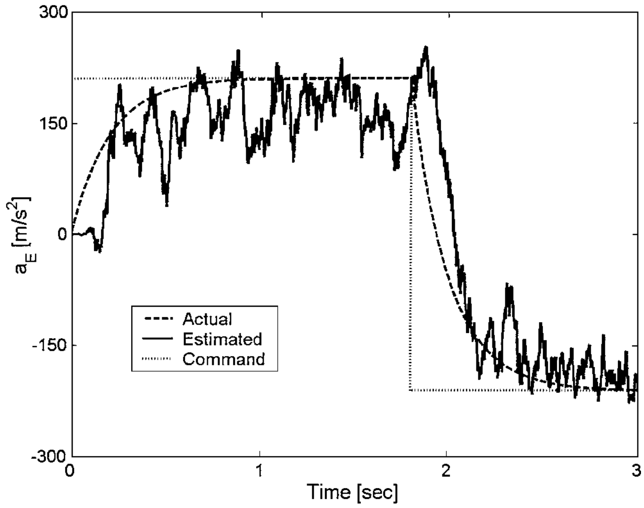

Fig. 5 Estimator performance.

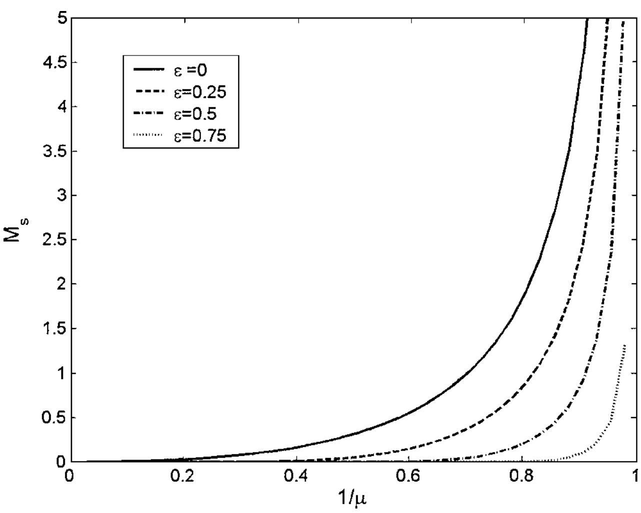

Fig. 4 Guaranteed normalized miss distance $M_s(\mu, \varepsilon)$.

When the solution (with the guidance law DGL/1) is characterized, two cases can be distinguished based on the value of $\mu\varepsilon$. This product has the physical interpretation of the pursuer/evader agility ratio. If the inequality $\mu\varepsilon < 1$ is satisfied, neither θ_s nor M_s (the guaranteed normalized miss distance for D_0) is zero. This includes, as a particular case, the game model of ideal evader dynamics ($\tau_E = \varepsilon = 0$) solved first by Gutman.[10] In this particular case, the term multiplying the actual lateral acceleration of the evader in Eq. (18) vanishes, and the implementation of the guidance law (DGL/0) becomes simpler. There is no need to know the maneuver of the evader because in this case it is a control variable. However, if $\mu\varepsilon \geq 1$, both θ_s and the corresponding M_s are zero. This means that, for the majority of all practical important initial conditions, a point capture of the evader is guaranteed against any feasible evasive maneuver. This robust hit-to-kill accuracy (an extremely desirable outcome in any interception scenario, particularly in antiballistic missile defense) is based on the assumption of perfect information, that is, accurate knowledge of all of the state variables, including the actual lateral acceleration of the evader. Unfortunately, this result remains only theoretical because in reality the guidance law is implemented in a noise-corrupted environment, rendering the information structure imperfect.

B. Implementation with Noisy Measurements

Testing the homing performance of the guidance laws DGL/0 and DGL/1, implemented with an estimator in a noise-corrupted environment, yields very disappointing results. Such a test, based on Monte Carlo simulations with different noise samples, was recently performed and reported.[18] Because the maneuvers of the antisurface missile are expected to be of a randomly switched bang–bang type, such maneuvers were used in the simulations. The simulation results demonstrated that these guidance laws, driven by the output of a Kalman-filter-type estimator, fail to provide a satisfactory homing performance. The robust hit-to-kill accuracy is lost, and the resulting miss distances are much larger than the values predicted by the perfect information game solution.

State variable estimation from noisy measurements for the implementation of a perfect information guidance law creates two types of errors: a (converged) steady-state estimation error and a delay. Because of the measurement noise, the estimated state, which in the present case is the normalized zero-effort miss distance defined by Eq. (18), is never the actual one. If the actual state variables are constant or vary slowly, the estimation error converges to a rather small value. The Kalman filter has the role to minimize the root mean square of this estimation error. However, if there is a sudden change in one of the variables, the estimation error of this variable becomes large, and it may take a substantial time until the estimated state converges to its new value. This phenomenon is clearly seen in a sample run shown in Fig. 5 .

C. Estimation Process Modeling

It has been a common experience that, even if the accuracy and the convergence of a position estimate are satisfactory, the estimated acceleration is less precise and it converges more slowly. Because this dynamic effect, seen in Fig. 5, is dominant, it is assumed in this study that the estimation of the variables x_1, x_2, and x_4 is ideal, whereas the estimation process of x_3 (the evader's lateral acceleration) is approximated by a perfect outcome delayed by the amount of Δt_{est}. There is a lower bound for the value of this delay, independent of the form of the estimator.[20] When this simplified model of the estimation process is used, a deterministic analysis could be carried out.[18,19]

As a consequence of the estimation delay, a maneuvering evader can generate a nonzero miss distance even if $\mu\varepsilon \geq 1$ because DGL/1 is the optimal guidance law only for the perfect information game ($\Delta t_{\text{est}} = 0$). For each value of $\Delta t_{\text{est}} > 0$, there exists an optimal evader maneuver that maximizes the miss distance. This maneuver consists of an optimally timed direction change (switch) of the maximal lateral acceleration command from left to right, or vice versa. An example for such a maneuver sequence is presented in Fig. 6, where the direction change is at $t = t_{sw}$. In Fig. 7, the maximum normalized miss distance generated by the evader is plotted as a function of the normalized estimation delay ($\Delta\theta_{\text{est}} = \Delta t_{\text{est}}/\tau_P$) for an example with $\mu = 2.25$ and $\varepsilon = 1$. The devastating performance degradation of DGL/1 due to the estimation delay is clearly observed. As expected, DGL/0 is not effected by the delay because it does not incorporate the evader's acceleration in the guidance law (assumes $\varepsilon = 0$, that is, ideal evader dynamics). From Fig. 7, it is evident that, for any

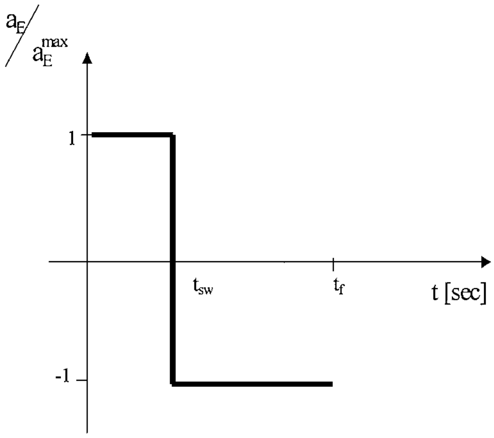

Fig. 6 Structure of the evasive maneuver.

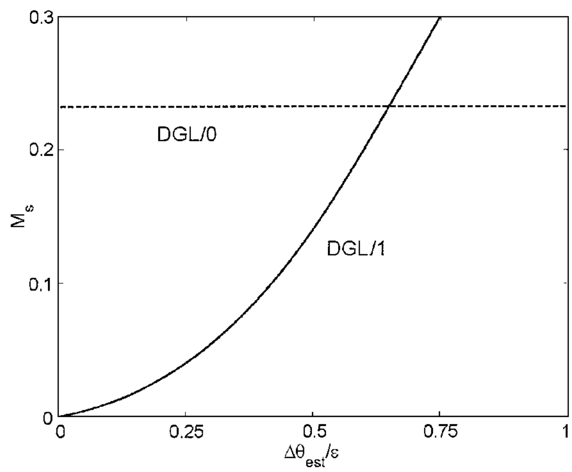

Fig. 7 Estimation delay effect: DGL/0 vs DGL/1.

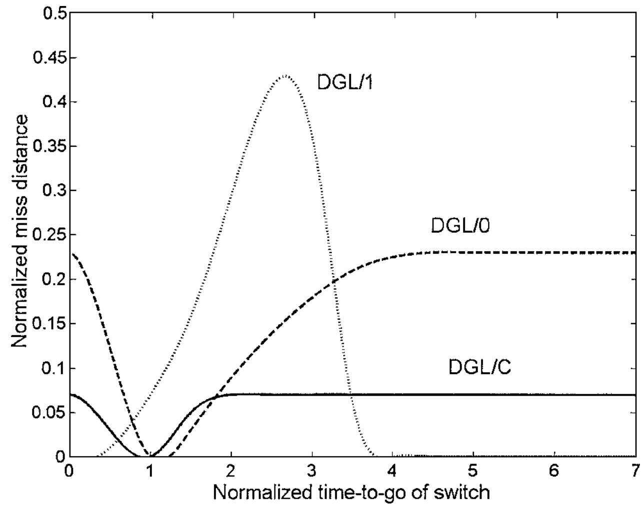

Fig. 8 Evasive maneuver effect; DGL/C compared to DGL/1 and DGL/0: μ = 2.25, ε = 1, and $\Delta\theta_{est}$ = 1.

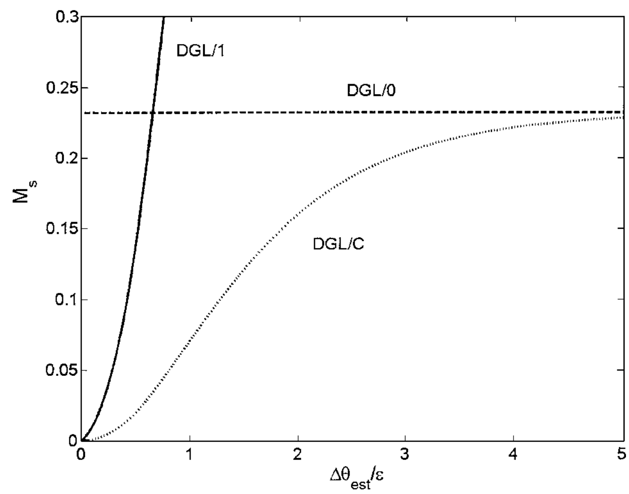

Fig. 9 Estimation delay effect; DGL/C compared to DGL/1 and DGL/0: μ = 2.25 and ε = 1.

amount of delay, a zero miss distance can no longer be achieved and that, if the estimation delay is too long, it is better not to incorporate the evader's acceleration in the guidance law at all, that is, to use DGL/0.

IV. Delayed Information Game Solution

A. New Problem Formulation

In this section a new guidance law (a modification of the original DGL/1), minimizing the effect of the estimation delay, is presented. It is intended to provide improved homing performance compared to both DGL/0 and DGL/1. For the derivation of the new guidance law, the interception scenario of a maneuvering antisurface missile is reformulated as a delayed information zero-sum pursuit–evasion game.

Given the dynamic system (5–8) and a set of initial conditions, solve the game with the cost function (12), subject to the following set of measurements available to the pursuer:

$$h_i(t) = x_i(t), \qquad i = 1, 2, 4, \qquad h_3(t) = x_3(t - \Delta t_{\text{est}}) \quad (27)$$

while the evader has perfect information on all of the state variables, as well as on the delay of the pursuer.

In an earlier paper,[18] an intuitive approach, inspired by the idea of reachable set,[21] was used. It suggested creating, based on the available information at every point of the time t, the reachable set of the evader and to aim the pursuer at the center of the convex hull of this reachable set. Simulation results, based on the implementation of this guidance law, denoted as DGL/C (compensated), demonstrated a substantial reduction of the maximum miss distance and the robustness with respect to the timing of the evader's maneuver as shown in Figs. 8 and 9 for an example with $\mu = 2.25$ and $\varepsilon = 1$.

At that time, an analytical solution was not yet available, and the guaranteed miss distance generated by the intuitively derived DGL/C could be predicted only by the simulations of the pure information delay model of the interception. In the following subsection, a recently developed rigorous analytical solution[19] of the delayed information zero-sum pursuit–evasion game formulation is presented, and an optimal guidance law based on its application is introduced.

B. Analytical Solution

The basic idea of the solution has been to replace the original normalized state variable $Z(\theta)$, defined in Eq. (18), by a new one, which is the center of the normalized uncertainty set (reachable set) created by the information delay. To enhance the physical understanding of the derivation process, in this first step the dimensional zero-effort miss distance $\boldsymbol{Z}(t)$ given in Eq. (16) will be used. This variable can be written as

$$z(t) = z^0(t) + \Delta z_E(t) \quad (28)$$

where

$$z^0(t) = x_1(t) + x_2(t)t_{\text{go}} - \Delta z_P(t) \quad (29)$$

$$\Delta z_P(t) = \tau_P^2 \psi(\theta) x_4(t) = \tau_P^2 \psi(t_{\text{go}}/\tau_P) x_4(t) \quad (30)$$

$$\Delta z_E(t) = \tau_E^2 \psi(\theta/\varepsilon) x_3(t) = \tau_E^2 \psi(t_{\text{go}}/\tau_E) x_3(t) \quad (31)$$

Because of the delayed measurement $h_3(t) = x_3(t - \Delta t_{\text{est}})$, the uncertain value of $x_3(t)$ is bounded by

$$[x_3(t)]_{\min} \le x_3(t) \le [x_3(t)]_{\max} \tag{32}$$

where the extreme values $[x_3(t)]_{\min}$ and $[x_3(t)]_{\max}$ are computed by integrating Eq. (7) with $a_E^c = -a_E^{\max}$ and $a_E^c = a_E^{\max}$, respectively,

$$[x_3(t)]_{\min} = x_3(t - \Delta t_{\text{est}}) \exp(-\Delta t_{\text{est}}/\tau_E) - a_E^{\max}[1 - \exp(-\Delta t_{\text{est}}/\tau_E)] \tag{33}$$

$$[x_3(t)]_{\max} = x_3(t - \Delta t_{\text{est}}) \exp(-\Delta t_{\text{est}}/\tau_E) + a_E^{\max}[1 - \exp(-\Delta t_{\text{est}}/\tau_E)] \tag{34}$$

The center of this segment of uncertainty is

$$[x_3(t)]^c \triangleq \{[x_3(t)]_{\max} + [x_3(t)]_{\min}\}/2 = x_3(t - \Delta t_{\text{est}}) \exp(-\Delta t_{\text{est}}/\tau_E) \tag{35}$$

The new variable, replacing $\boldsymbol{Z}(t)$, is defined as

$$z^c(t) \triangleq z^0(t) + \Delta z_E^c(t) \tag{36}$$

where

$$\Delta z_E^c(t) = \tau_E^2 \psi(t_{go}/\tau_E) x_3(t - \Delta t_{\text{est}}) \exp(-\Delta t_{\text{est}}/\tau_E) \tag{37}$$

The nondimensional form of Eq. (36), based on Eq. (18), is

$$Z^c(\theta) = Z(\theta) + \varepsilon^2 \psi(\theta/\varepsilon)[x_3(\theta + \Delta\theta_{\text{est}}) \times \exp(-\Delta\theta_{\text{est}}/\varepsilon) - x_3(\theta)] \big/ a_E^{\max} \tag{38}$$

where $\Delta\theta_{\text{est}} = \Delta t_{\text{est}}/\tau_P$.
The derivative of Eq. (38) with respect to θ is

$$\frac{\mathrm{d}Z^c}{\mathrm{d}\theta} = \frac{\mathrm{d}Z}{\mathrm{d}\theta} + \frac{\varepsilon^2}{(a_E)^{\max}} \left\{ \left[x_3(\theta + \Delta\theta_{\text{est}}) \exp\left(-\frac{\Delta\theta_{\text{est}}}{\varepsilon}\right) - x_3(\theta) \right] \times \frac{\mathrm{d}\psi(\theta/\varepsilon)}{\mathrm{d}\theta} + \psi\left(\frac{\theta}{\varepsilon}\right) \times \frac{\mathrm{d}}{\mathrm{d}\theta}\left[x_3(\theta + \Delta\theta_{\text{est}}) \exp\left(-\frac{\Delta\theta_{\text{est}}}{\varepsilon}\right) - x_3(\theta) \right] \right\} \tag{39}$$

where

$$\frac{\mathrm{d}\psi(\theta/\varepsilon)}{\mathrm{d}\theta} = \frac{1 - \exp(-\theta/\varepsilon)}{\varepsilon} \tag{40}$$

By integrating Eq. (7) between $(t - \Delta t_{\text{est}})$ and t, using Eq. (9), one gets

$$x_3(t) = x_3(t - \Delta t_{\text{est}}) \exp\left(-\frac{\Delta\theta_{\text{est}}}{\varepsilon}\right) + \frac{(a_E)^{\max}}{\tau_E} \int_{t - \Delta t_{\text{est}}}^{t} \exp\left(-\frac{t - s}{\tau_E}\right) \boldsymbol{v}(s)\, \mathrm{d}s \tag{41}$$

which can be rewritten, using the normalized time-to-go θ defined in Eq. (17), as

$$x_3(\theta + \Delta\theta_{\text{est}}) \exp\left(-\frac{\Delta\theta_{\text{est}}}{\varepsilon}\right) - x_3(\theta) = \frac{(a_E)^{\max}}{\varepsilon} \int_{\theta + \Delta\theta_{\text{est}}}^{\theta} \exp\left(\frac{\theta - \sigma}{\varepsilon}\right) \boldsymbol{v}(\sigma)\, \mathrm{d}\sigma \tag{42}$$

with $\sigma = (t_f - s)/\tau_P$. Consequently,

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left[x_3(\theta + \Delta\theta_{\text{est}}) \exp\left(-\frac{\Delta\theta_{\text{est}}}{\varepsilon}\right) - x_3(\theta) \right] = \frac{(a_E)^{\max}}{\varepsilon^2} \left\{ \varepsilon \left[\boldsymbol{v}(\theta) - \exp\left(-\frac{\Delta\theta_{\text{est}}}{\varepsilon}\right) \boldsymbol{v}(\theta + \Delta\theta_{\text{est}}) \right] + \int_{\theta + \Delta\theta_{\text{est}}}^{\theta} \exp\left(\frac{\theta - \sigma}{\varepsilon}\right) \boldsymbol{v}(\sigma)\, \mathrm{d}\sigma \right\} \tag{43}$$

Substituting Eqs. (21), (42), and (43) into Eq. (39), setting $\rho = \sigma - \theta$, and eliminating equal terms leads to the expression

$$\frac{\mathrm{d}Z^c}{\mathrm{d}\theta} = \mu\psi(\theta)\boldsymbol{u}(\theta) - \varepsilon\psi\left(\frac{\theta}{\varepsilon}\right) \exp\left(-\frac{\Delta\theta_{\text{est}}}{\varepsilon}\right) \boldsymbol{v}(\theta + \Delta\theta_{\text{est}}) + \left(\frac{\theta}{\varepsilon}\right) \int_{\Delta\theta_{\text{est}}}^{0} \exp\left(-\frac{\rho}{\varepsilon}\right) \boldsymbol{v}(\rho)\, \mathrm{d}\rho \tag{44}$$

Because the evader also knows Z^c, Eq. (44) represents the dynamics of a perfect information game with delayed control. When the definition of Z^c in Eq. (38) is noted and that $\psi(\alpha = 0) = 0$ from Eq. (19) is kept in mind, the cost function of this game is identical to the cost of the original game in Eq. (22) with the same bounded normalized controls of Eqs. (8) and (9). The mathematical procedure leading to the solution of this game, detailed in Ref. 19, is out of the scope of the present paper. Here only end results are repeated. The candidate optimal strategies, satisfying the necessary conditions, are

$$\boldsymbol{u}^*(\theta) = \boldsymbol{v}^*(\theta) = \mathrm{sign}\{Z^c(\theta)\} \tag{45}$$

confirming the validity of the intuitively derived DGL/C.[18] Substituting Eq. (45) into Eq. (44) and carrying out the integration leads to expressing the optimal game dynamics as

$$\frac{\mathrm{d}Z^c}{\mathrm{d}\theta} = \Gamma^c(\theta, \Delta\theta_{\text{est}})\, \mathrm{sign}\{Z^c(\theta)\} \tag{46}$$

where

$$\Gamma^c(\theta, \Delta\theta_{\text{est}}) = \mu\psi(\theta) - \varepsilon\psi(\theta/\varepsilon) \exp(-\Delta\theta_{\text{est}}/\varepsilon) + \theta[\exp(-\Delta\theta_{\text{est}}/\varepsilon) - 1] \tag{47}$$

Equation (46) is similar to the form of Eq. (24) obtained for the solution of the perfect information games as DGL/0 and DGL/1. It means that the decomposition of the normalized game space (Z^c, θ) has the same structure as Fig. 2. Similarly, θ_s^c is the nonvanishing solution $(\theta \neq 0)$ of

$$\Gamma^c(\theta, \Delta\theta_{\text{est}}) = 0 \tag{48}$$

and M_s^c can be computed by integrating $\Gamma^c(\theta, \Delta\theta_{\text{est}})$ between $\theta = 0$ and θ_s^c. The values of θ_s^c and M_s^c are plotted as the function of $(\Delta\theta_{\text{est}}/\varepsilon)$ in Figs. 10 and 11, respectively, for $\mu = 2.25$ and $\varepsilon = (0, 0.25, 0.5, 1.0)$.

The analysis[19] indicates that this parameter

$$\delta \triangleq \Delta\theta_{\text{est}}/\varepsilon = \Delta t_{\text{est}}/\tau_E \tag{49}$$

is governing the delayed information game solution. When Eq. (47) is compared to Eq. (25), it can be directly observed that, for all $\Delta\theta_{\text{est}} > 0$, one always has $\theta_s^c > \theta_s$ and consequently $M_s^c > M_s$. It means that due to the estimation delay the guaranteed miss distance can never be zero, as it is clearly seen in Figs. 10 and 11.

V. Monte Carlo Simulations

A. Estimator Selection

As noted in Sec. III, the implementation of a guidance law in a noise corrupted environment requires an estimator. In this paper, two different types of discrete time estimators (with time step of Δt) were used. The first one was a classical Kalman filter, where the evader's random maneuver was represented as a stochastic process in the form of a random telegraph (RT) signal, characterized by a single parameter λ (the average frequency of the signal). For the estimator design, such a process was represented by white noise

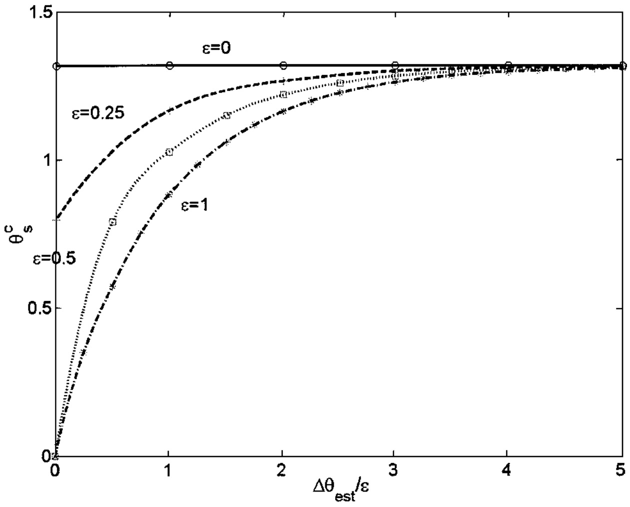

Fig. 10 $\theta_s^c\ (\mu, \varepsilon)$ **as the function of** $(\Delta\theta_{\text{est}}/\varepsilon)$**:** $\mu = 2.25$ **and** $\varepsilon = (0, 0.25, 0.5, 1.0)$**.**

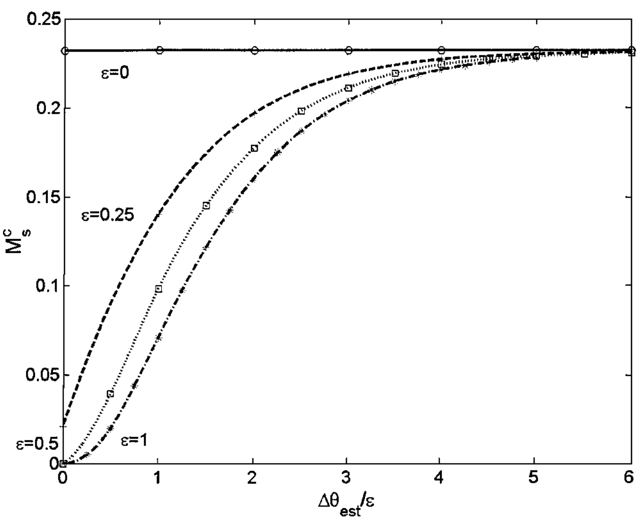

Fig. 11 $M_s(\mu, \varepsilon)$ **as the function of** $(\Delta\theta_{\text{est}}/\varepsilon)$**:** $\mu = 2.25$ **and** $\varepsilon = (0, 0.25, 0.5, 1.0)$**.**

going through a first-order shaping filter[22] with a time constant of $1/(2\lambda)$ because both have the same autocorrelation function. The well-known equations of this estimator can be found in the technical literature[23] and, therefore, are not repeated here. The second estimator was a fast multiple model adaptive estimator (MMAE), described in detail (out of the scope of this paper) in Ref. 24. The fast MMAE uses several target maneuver models and has a potential of improved estimation if the number of models is sufficiently high. The required computational effort is only slightly higher than needed for a classical Kalman filter with shaping filter (KF/SF). In this paper the results of using the fast MMAE, applying the minimum mean square error (MMSE) weighting method, are compared to the results obtained when using the KF/SF. The MMAE used incorporates 30 models, each corresponding to a different timing of the bang–bang maneuver switch in the end game. For both estimators, it is assumed that the interceptor acquires measurements at a given frequency f. The measurement vector is

$$\boldsymbol{Y} = \begin{bmatrix} \phi_{az} \\ a_P \end{bmatrix} + \begin{bmatrix} \nu_{az} \\ \nu_{ap} \end{bmatrix} \cong \begin{bmatrix} 1/r & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \nu_{az} \\ \nu_{ap} \end{bmatrix} \tag{50}$$

where r is the accurately measured range (Fig. 1), ν_{az} is the angular noise of the seeker, and ν_{ap} is the measurement noise of the interceptor's acceleration. The noise statistics are given by

$$v_i \sim WN\left(0, \sigma_i^2\right), \qquad i = az, ap \tag{51}$$

Table 1 Simulation parameters

Parameter	Value	Remarks
$a_E^{\max}$	21.5 g	
$a_P^{\max}$	48.4 g	$\Rightarrow \mu = 2.25$
τ_E	0.2 s	
τ_P	0.2 s	$\Rightarrow \varepsilon = 1$
σ_{az}	0.2 mrad	
σ_{ap}	0.1 g	
V_c	5.5 km/s	
R_0	16.5 km	$\Rightarrow t_f = 3$ s
Δt	0.001 s	
f	500 Hz	
λ	1.5 1/s	

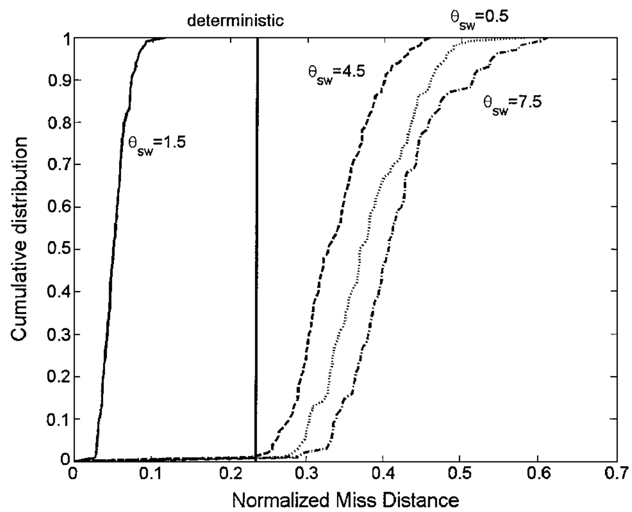

Fig. 12 DGL/0 homing performance.

B. Simulation Parameters

In simulating the implementation of the guidance laws DGL/0, DGL/1, and DGL/C, sets of 100 Monte Carlo runs with different noise samples were used. To allow comparison between the different cases investigated, the same random number generator seed has been selected for each set of the 100 Monte Carlo runs. The simulation parameters are summarized in Table 1.

C. Worst-Case Evasive Maneuvers

Because the maneuvers of the antisurface missile are expected to be of a randomly switched bang–bang type (Fig. 6), the worst maneuver timing (the one that creates the largest miss distance) has to be identified. The accumulated normalized miss distance distributions obtained by the bang–bang maneuvers with different switching times are shown for DGL/0 and DGL/1 with KF/SF in Figs. 12 and 13, respectively. In Fig. 12, the guaranteed miss distance predicted by the perfect information game solution is also indicated. It is evident that for most of the target maneuver timings, presented in Fig. 12, the miss distances are larger than the deterministic prediction based on the value of the perfect information game. The only exception is the maneuver with $\theta_{sw} = 1.5$ because in this case the timing of the evasive maneuver is clearly nonoptimal. In this case, the nonoptimal timing outweighs the effect of the estimation delay. From Figs. 12 and 13, the timing of the worst case maneuvers can also be identified as $\theta_{sw} = 7.5$ for DGL/0 and $\theta_{sw} = 2$ for DGL/1. These results are similar (although not identical) to those obtained in Ref. 18, where a different estimator was used.

The present paper concentrates on the homing performance of the new guidance law, denoted as DGL/C, in a noise-corrupted scenario. It turns out that the worst maneuver timing for DGL/C depends on the value of the assumed estimation delay to be compensated. This observation lead to performing a simultaneous min–max search for the best estimation delay and the worst timing of the evader maneuver. The accumulated normalized miss distance distribution

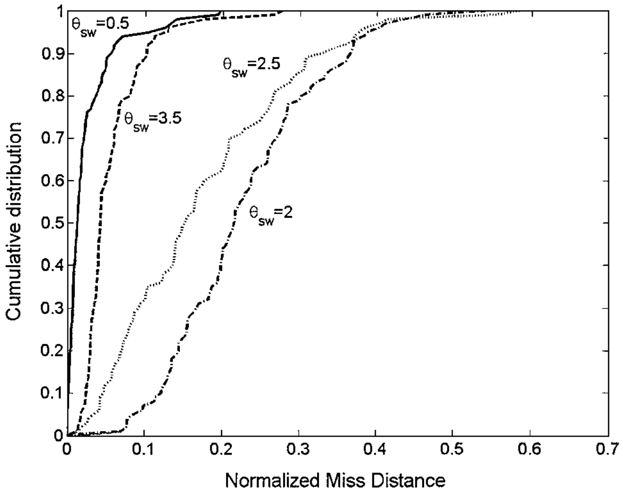

Fig. 13 DGL/1 homing performance.

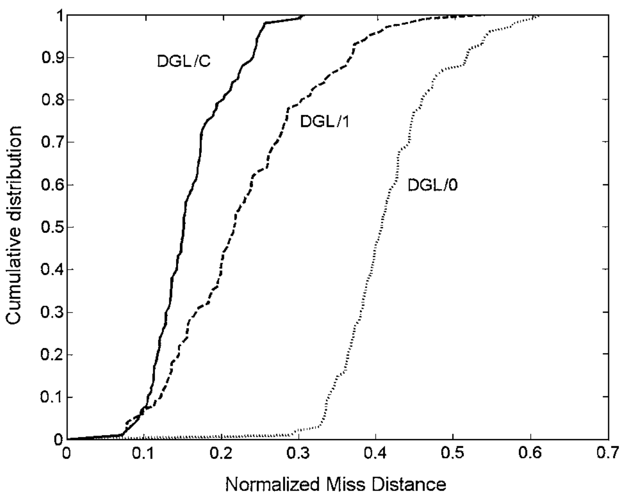

Fig. 14 Cumulative miss distance distribution comparison, KF/SF.

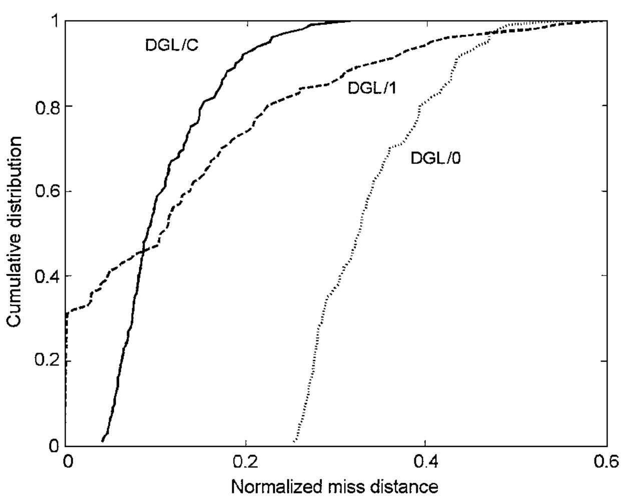

Fig. 15 Cumulative miss distance distribution comparison, MMAE/MMSE.

obtained for the optimal pair (the best estimation delay is $\Delta\theta_{\text{est}} = 0.5$, whereas the worst evader maneuver timing is $\theta_{sw} = 7.5$) is shown in Fig. 14 using a KF/SF estimator and is compared with the worst case results of the Figs. 12 and 13. Similar results obtained using the MMAE/MMSE estimator are presented in Fig. 15. Figures 14 and 15 show a substantial improvement achieved by applying the new delay-compensating guidance law. The accumulated normalized miss distance distribution obtained for DGL/C by the two different estimators is compared in Fig. 16.

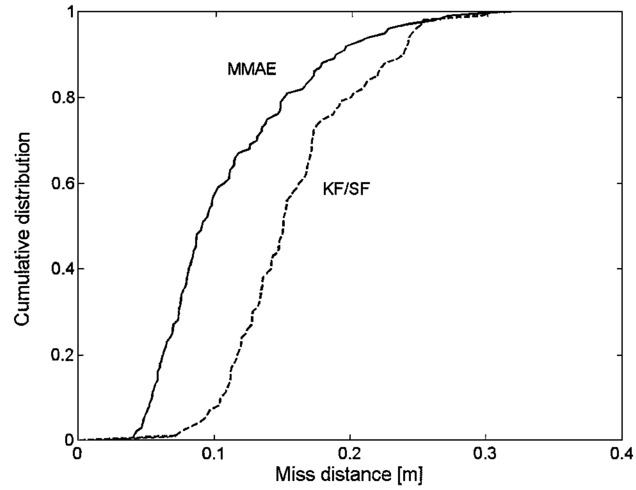

Fig. 16 DGL/C homing performance.

D. Discussion

As can be seen in Fig. 12, the miss distances for DGL/0 in the noise-corrupted environment are of the same order of magnitude (though larger) as the deterministic prediction. Moreover, the worst homing performance, represented by the accumulated normalized miss distance distributions, seems to be less affected by the timing of the random evasive maneuver because this guidance law does not use the information on the evader's lateral acceleration. The performance degradation of DGL/1 due to the noise is much more dramatic, as seen in Fig. 13. Instead of the guaranteed zero miss distance, predicted by the perfect information game solution, the worst evader maneuver timing generates substantial miss distances. The high sensitivity to the timing of the random evasive maneuver is also observed.

In spite of these qualitative observations, it appears from Fig. 14 that (at the noise level used in the simulation) DGL/1 has a slightly smaller maximum miss distance and a more favorable distribution than DGL/0. The reason is the relatively small estimation delay ($\Delta\theta_{\text{est}} = 0.5$) associated with the noise level of $\sigma_{az} = 0.2$ mrad, as mentioned earlier. For a larger noise level, the situation could be different, as indicated in Ref. 24.

The simulation results, presented in Fig. 14, clearly confirm the improved homing performance of DGL/C compared to the other guidance laws, as predicted by the deterministic analysis in Secs. III and IV and shown in Fig. 9. The maximum miss distance (compared to DGL/1) is reduced to less than half, and the entire distribution becomes more favorable. For example, given a normalized lethality radius of 0.2, DGL/C provides a probability of success of about 0.80, compared to a probability of 0.35 for DGL/1. Moreover, the sensitivity to the timing of the random evasive maneuver is also reduced (similarly to DGL/0). When Figs. 14 and 15 are compared, the different performance of the two estimators is clearly observed. From Fig. 16, one can learn that although MMAE/MMSE has a more favorable miss distance distribution than the KF/SF, the maximum miss distance in both cases is similar.

VI. Conclusions

A new approach for guidance law development for the interception of maneuvering targets is described. The scenario is modeled as a zero-sum pursuit–evasion game. The new approach takes into account the inherent delay of the estimation process in a noise-corrupted environment and attempts to compensate for it. The paper presents the application of the rigorous analytical solution of a pursuit–evasion game with delayed information of the pursuer, modeling the noise-corrupted interception scenario. The analytical solution confirms the validity of an earlier intuitively derived guidance law, partially compensating for the pursuer's information delay created by the estimation process.

This nonorthodox approach has generated for the first time in the open literature (to the best of the authors' knowledge) a guidance law that has been derived by explicitly considering the outcome of the estimation process. The new guidance law, obtained by a deterministic analysis, was tested by a large set of Monte Carlo simulations in a generic but realistic noise corrupted scenario, using two types of estimators, confirming its usefulness. The worst-case miss distances, as well as the sensitivity to the timing of the random evasive maneuvers, are substantially reduced. These results represent a potential breakthrough in guidance law design. The two elements of the innovative mindset, namely, the pursuit–evasion game formulation and not following the (commonly used) unjustified application of the certainty equivalence property, project a hope that small miss distances, compatible with a hit-to-kill requirement, can be achieved even in highly stressed interception scenarios expected in the future.

Note that the implementation of the delay-compensating guidance law does not require any specific hardware, and as a consequence, its insertion into any existing missile defense system can be easily carried out.

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